

〈Paper〉

〈論文〉

Global Dimension of Algebras Whose Quiver Contain Two Oriented Cycles

有向サイクルを2つ持つ多元環の大局次元

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Let A be a finite dimensional basic connected algebra over an algebraically closed field k , and n is the number of the non isomorphic simple left modules of A . If A is quasi-hereditary, then the global dimension $gl.dim.A$ is less than or equal to $2n - 2$, and the Loewy length $l(A)$ of A is less than or equal to $2^n - 1$ [3]. In the previous paper, we show that serial quasi-hereditary algebras have global dimension less than or equal to n , and $l(A)$ is less than or equal to $2n - 1$ [4].

In this note we show that the algebra whose quiver contains unique oriented cycle has global dimension less than or equal to $2n - 2$ if it has finite global dimension. Moreover, in case of quasi-hereditary algebras, we compute the global dimension of an algebra whose quiver contains some essential oriented cycles.

1. THE ALGEBRA WHOSE QUIVER CONTAINS AT MOST ONE ORIENTED CYCLE

First we fix some notations. Let Q_A be the quiver of A , $\{1, 2, \dots, n\}$ be its vertices, and $\{e_1, \dots, e_n\}$ be the set of corresponding primitive idempotents of A .

A vertex i of Q_A is called a sink(resp., source) vertex if all arrows which contain i have i as ending(resp., starting) point. If vertex i is a sink vertex, then Ae_i is a simple projective module. Dually, if vertex i is a source vertex, then e_iA is a simple injective module.

Lemma 1. *Suppose that Q_A has a sink or source vertex, that is there is a primitive idempotent e of A with simple projective module Ae or simple injective module eA . Let $B = A/AeA$. Then*

$gl.dim.A \leq gl.dim.B + 1,$
 $l(A) \leq l(B) + 1,$
and these bounds are sharp.

Proof. The second inequality is trivial. For the first, assume that Ae is simple projective. Since AeA is direct sum of Ae , ${}_AAeA$ is projective as left A -module. Let ${}_AX$ be an arbitrary left A -module. If ${}_AAeAX \neq 0$, $AeAX$ is direct sum of Ae and projective. Hence, $proj.dim.{}_AX \leq proj.dim.{}_A(X/AeAX)$. If $AeAX = 0$, ${}_AX$ is a B -module. It is enough to show that if X is a B -module, then

$$proj.dim.{}_AX \leq 1 + proj.dim.{}_BX.$$

This is the same as the Statement 1 of [3].

In case of eA is simple injective, we can show by duality. □

Remark 2. In the above Lemma, the word “simple” can be replaced by “semi simple”.

Next theorem is due to Gustafson[2].

Theorem 3 (Gustafson). *Let A be a serial algebra. If the global dimension of A is finite, then*

$gl.dim.A \leq 2n - 2,$
 $l(A) \leq 2n - 1,$
and these bounds are sharp.

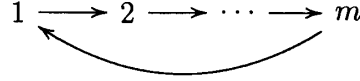
Serial algebras have at most one oriented cycle. So, next theorem is generalization of Theorem 3.

Theorem 4. *Let A be an algebra whose quiver contains at most one oriented cycle. If the global dimension of A is finite, then*

$gl.dim.A \leq 2n - 2,$
 $l(A) \leq 2n - 1,$
and these bounds are sharp.

Proof. If Q_A has no oriented cycle, nothing has to be proved. This is the Lemma 1 of [4]. We may assume Q_A has unique oriented cycle. If Q_A has a sink or source

vertex, take the corresponding idempotent e and make new algebra $B = A/AeA$. Then Q_B contains one oriented cycle and by the Lemma 1 $gl.dim.A \leq gl.dim.B + 1$. We continue this process until we get the algebra U that has no simple projective nor simple injective module. Then Q_U is the following shape.



U must be a serial algebra. So the Theorem 3, $gl.dim.U \leq 2m - 2$ and $l(U) \leq 2m - 1$ where m is the number of the non isomorphic simple left modules of U . Using Lemma 1 repeatedly, we can prove the theorem. For the sharpness of bounds, these are the same as the Gustafson's Example[2]. \square

If A is a left or right (not necessarily both) serial algebra and has finite global dimension, the condition of the theorem holds. So the following corollary is the direct consequence of the Theorem 4.

Corollary 5. *Let A be a left or right serial algebra. If the global dimension of A is finite, then*

$$gl.dim.A \leq 2n - 2,$$

$$l(A) \leq 2n - 1,$$

and these bounds are sharp.

More over if one of the equality holds, A is a serial algebra.

2. QUASI-HEREDITARY ALGEBRAS WITH SOME ORIENTED CYCLES

Let N be the Jacobson radical of A . An idempotent e is said to be a heredity idempotent of A if $eNe = 0$ and AeA is projective as a right A -module.

Let $\{e_1, e_2, \dots, e_n\}$ be fixed ordering of the complete set of primitive orthogonal idempotents of A . An algebra A is said to be quasi-hereditary algebra with respect to this ordering if for any $1 \leq t \leq n$, $\bar{\epsilon}_t$ is a heredity idempotent of $A/A\epsilon_{t+1}A$, where $\epsilon_j = e_j + e_{j+1} + \dots + e_n$ for $1 \leq j \leq n$ and $\epsilon_{n+1} = 0$. Such a sequence called a heredity sequence.

The following lemma is due to Dlab and Ringel[3].

Lemma 6. Suppose that A has a heredity idempotent e . Let $B = A/AeA$. Then
 $gl.dim.A \leq gl.dim.B + 2$
 $l(A) \leq 2l(B) + 1$.

We call an oriented cycle is essential when it is made by the distinct vertices.

Theorem 7. Let A be a quasi-hereditary algebra with m essential oriented cycles. Then

$$gl.dim.A \leq n + m - 1,$$

$$l(A) \leq 2^m(n - m + 1),$$

and these bounds are sharp.

Proof. If Ae_n is simple projective or e_nA is simple injective, we use Lemma 1 and consider the algebra $B = A/Ae_nA$. Then B is again quasi-hereditary algebra with respect to $\{e_1, e_2, \dots, e_{n-1}\}$ which has same number of essential oriented cycles. Otherwise, if n is not sink nor source vertex, then n is belong to some essential oriented cycle. In this case, we use Lemma 6 and consider the algebra $B = A/Ae_nA$. B is again quasi-hereditary algebra with respect to $\{e_1, e_2, \dots, e_{n-1}\}$ that has $m - 1$ or less essential oriented cycles.

We continue these processes. Second case does occur at most m times. So we conclude that

$$gl.dim.A \leq 2m + (n - m) = n + m - 1,$$

$$l(A) \leq 2^m(n - m + 1).$$

□

Remark 8. In the above theorem, we may assume $m \leq n - 1$. Since otherwise these bounds exceed $2n - 2$ and $2^n - 1$ respectively.

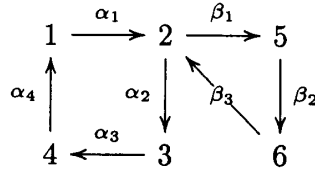
Example 9. Let A be the algebra of

$$\begin{array}{ccccccc} 1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \xrightarrow{\alpha_3} & 4 \xrightarrow{\alpha_4} 5 \\ & \xleftarrow{\beta_1} & & \xleftarrow{\beta_2} & & \xleftarrow{\beta_3} & \end{array}$$

with relation $\{\alpha_4\alpha_3 = \alpha_3\alpha_2 = \alpha_2\alpha_1 = \beta_3\beta_2 = \beta_2\beta_1 = 0, \alpha_1\beta_1 = \beta_2\alpha_2, \alpha_2\beta_2 = \beta_3\alpha_3\}$. Then A is a quasi-hereditary algebra with respect to $\{e_1, e_2, e_3, e_4, e_5\}$ and has 3

essential oriented cycles. Ae_5 is simple projective module. This is the case of $n = 5$ and $m = 3$ in Theorem 7, and $gl.dim.A = 7$. This shows that the bound of global dimension is sharp.

Example 10. Let A be the algebra of



with relation $\{\alpha_4\alpha_3 = \alpha_3\alpha_2 = \alpha_1\alpha_4 = 0, \beta_2\beta_1 = \beta_1\beta_3 = 0, \beta_1\alpha_1 = \alpha_2\beta_3 = 0\}$. Then A is a quasi-hereditary algebra with respect to $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and has 2 essential oriented cycles. Ae_6 is not simple. Projective dimension of S_6 is 7. Indeed,

$$P_6 \longrightarrow P_5 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_4 \oplus P_6 \longrightarrow P_3 \oplus P_5 \longrightarrow P_2 \longrightarrow P_6 \longrightarrow S_6 \longrightarrow 0$$

is the minimal projective resolution of S_6 . Where P_i and S_i are corresponding indecomposable projective module Ae_i and simple left module Ae_i/Ne_i respectively. This is maximal among S_i , so $gl.dim.A = 7$. This is the case of $n = 6$ and $m = 2$ in Theorem 7, and $gl.dim.A = 7 = 6 + 2 - 1$ is maximal.

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